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CERTAIN GENERALIZATIONS OF ENESTRÖM-KAKEYA THEOREM

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According to well known Eneström-Kakeya theorem, the polynomial $p(z) = \Sigma_{j=0}^{n} a_j z^j$ has all its zeros in $|z| \leq 1$, provided

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.$$

We have obtained certain generalizations of this theorem for the polynomial $p(z) = \Sigma_{j=0}^{n} a_j z^j$, (Re $a_j = \alpha_j$, Im $a_j = \beta_j$), with coefficients such that

(i) a_j's $(0 \leq j \leq n)$are real and for certain non-negative real numbers t_1, t_2 $(t_1 \geq t_2$ & $t_1 \neq 0)$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0 \quad (r = 1, 2, \ldots, n+1)$$

$$a_{-1} = a_{n+1} = 0.$$

or

(ii) For certain non-negative integers $k_1, k_2, \ldots, k_p; r_1, r_2, \ldots, r_q$ and for certain $t\ (> 0)$

$$\alpha_0 \leq t\alpha_1 \leq \ldots \leq t^{k_1}\alpha_{k_1} \geq t^{k_1+1}\alpha_{k_1+1} \geq \ldots \geq t^{k_2}\alpha_{k_2} \leq t^{k_2+1}\alpha_{k_2+1} \leq \ldots \quad (1)$$

$$\beta_0 \leq t\beta_1 \leq \ldots \leq t^{r_1}\beta_{r_1} \geq t^{r_1+1}\beta_{r_1+1} \geq \ldots \geq t^{r_2}\beta_{r_2} \leq t^{r_2+1}\beta_{r_2+1} \leq \ldots \quad (2)$$

(with inequalities, getting reversed at p indices $k_1, k_2, \ldots, k_p$, in (1) and $t^n\alpha_n$, being the last term in (1), and similarly, inequalities getting reversed at q indices $r_1, r_2, \ldots, r_q$ in (2) and $t^n\beta_n$, being the last term in (2)).

or

(iii) For certain real β

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2 \quad (j = 0, 1, \ldots, n)$$

and for certain non-negative integers $k_1, k_2, \ldots, k_p$ and for certain $t\ (> 0)$

$$|a_0| \leq t|a_1| \leq \ldots \leq t^{k_1}|a_{k_1}| \geq t^{k_1+1}|a_{k_1+1}| \geq \ldots \geq t^{k_2}|a_{k_2}| \leq t^{k_2+1}|a_{k_2+1}| \leq \ldots \quad (3)$$

(with inequalities getting reversed at p indices $k_1, k_2, \ldots, k_p$, in (3) and $t^n|a_n|$, being the last term in (3))

or

(iv) for certain real β,

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2 \quad (j = 0, 1, \ldots, n)$$

and for certain $t\ (> 0)$ and $k\ (0 \leq k \leq n)$

$$t^n|\alpha_n| \leq t^{n-1}|\alpha_{n-1}| \leq \ldots \leq t^k|\alpha_k| \geq t^{k-1}|\alpha_{k-1}| \geq \ldots \geq |\alpha_0|$$

1. Introduction and Statement of Results

The following result is well known in the theory of the distribution of zeros of polynomials.

THEOREM A (Eneström-Kakeya). *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq a_{n-2} \ldots \geq a_1 \geq a_0 > 0,$$

then all its zeros lie in $|z| \leq 1$.

There already exist in literature [1–15], certain generalizations and refinements of Eneström-Kakeya theorem. Joyal *et al.* [13] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

THEOREM B. *If*

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_2 \geq a_1 \geq a_0,$$

then the polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ *has all its zeros in the disc*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Aziz and Mohammad [1] obtained the following generalization of Theorem A.

THEOREM C. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n, with positive coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad r = 1, 2, \ldots n+1, \quad (a_{-1} = a_{n+1} = 0),$$

then all zeros of $p(z)$ *lie in* $|z| \leq t_1$.

We have obtained a generalization of Theorem C, by following the direction of generalization of Theorem A to Theorem B and also a refinement of Theorem C. More precisely, we have proved

THEOREM 1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ $(a_0 \neq 0)$, *be a polynomial of degree n, with real coefficients such that for certain non-negative real numbers* t_1, t_2 $(t_1 \geq t_2$ and $t_1 \neq 0)$,

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad (r = 1, 2, \ldots, n+1), \tag{1.1}$$

$$a_{-1} = a_{n+1} = 0. \tag{1.2}$$

Then $p(z)$ *has all its zeros in*

$$\min\left\{\frac{|a_0| t_1 t_2}{|a_0| t_1^{n+1} + a_n t_1^{n+1} - a_0 t_2}, t_1\right\} \leq |z| \leq \max\left\{\frac{a_n t_1 - \dfrac{a_0 t_2}{t_1^n} + \dfrac{|a_0| t_2}{t_1^n}}{|a_n|}, t_1\right\}. \tag{1.3}$$

Thinking again in terms of Theorem B, but indirectly, with its following generalization obtained by Gardner and Govil [6],

THEOREM D. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n. If Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, *and for some k and r and for some* $t\ (>0)$,

$$\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \ldots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \ldots \geq t^n\alpha_n, \tag{1.4}$$

and

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \ldots \leq t^r\beta_r \geq t^{r+1}\beta_{r+1} \geq \ldots \geq t^n\beta_n, \tag{1.5}$$

then $p(z)$ *has all its zeros in* $R_1 \leq |z| \leq R_2$, *where*

$$R_1 = \min\{(t|a_0|/(2(t^k\alpha_k + t^r\beta_r) - (\alpha_0 + \beta_0) - t^n(\alpha_n + \beta_n - |a_n|)), t\}$$

$$\begin{aligned} R_2 = \max\{(|a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n) + (t^2+1)(t^{n-k-1}\alpha_k \\ + t^{n-r-1}\beta_r) + (t^2 - 1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) \\ + (1 - t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right))/|a_n|, 1/t\}, \end{aligned}$$

we have obtained, by making inequalities in (1.4) and (1.5), getting reversed at p indices and q indices respectively, the following generalization of Theorem D:

THEOREM 2. *Let* $p(z) = \sum_{j=0}^{n} a_n z^j$, *be a polynomial of degree n. If Re* $a_j = \alpha_j$, *Im* $a_j = \beta_j$, *for* $j = 0, 1, 2, \ldots, n$ *and for certain non-negative integers* $k_1, k_2, \ldots, k_p$; $r_1, r_2, \ldots, r_q$ *and for certain* $t > 0$

$$\alpha_0 \leq t\alpha_1 \leq \ldots \leq t^{k_1}\alpha_{k_1} \geq t^{k_1+1}\alpha_{k_1+1} \geq \ldots \geq t^{k_2}\alpha_{k_2} \leq t^{k_2+1}\alpha_{k_2+1} \leq \ldots, \tag{1.6}$$

$$\beta_0 \leq t\beta_1 \leq \ldots \leq t^{r_1}\beta_{r_1} \geq t^{r_1+1}\beta_{r_1+1} \geq \ldots \geq t^{r_2}\beta_{r_2} \leq t^{r_2+1}\beta_{r_2+1} \leq \ldots, \tag{1.7}$$

(with inequalities getting reversed at p indices $k_1, k_2, \ldots, k_p$ *in (1.6) and* $t^n\alpha_n$, *being the last term in (1.6), and similarly, inequalities getting reversed at q indices* $r_1, r_2, \ldots, r_q$ *in (1.7) and* $t^n\beta_n$ *being the last term in (1.7)), then all zeros of* $p(z)$ *lie in*

$$R_1 \leq |z| \leq R_2,$$

where

$$R_1 = min\left(\frac{t^2|a_0|}{M_1}, t\right) = \frac{t^2|a_0|}{M_1} = \frac{t^2|a_0|}{M_1'}$$

$$R_2 = max\left(\frac{M_2}{|a_n|}, \frac{1}{t}\right),$$

$$M_1 = tM_1',$$

$$M_1' = -\{\alpha_0 + (-1)^{p+1}\alpha_n t^n + \sum_{u=1}^{p} (-1)^u \alpha_{k_u} t^{k_u}\}$$
$$- \{\beta_0 + (-1)^{q+1}\beta_n t^n + \sum_{s=1}^{q} (-1)^s \beta_{r_s} t^{r_s}\} + |a_n| t^n,$$

$$M_2 = [-\alpha_0 t^{n-1} + (-1)^{p+1}\alpha_n t + (t^2+1) \sum_{u=1}^{p} (-1)^u \alpha_{k_u} t^{n-k_u-1}$$
$$+ (t^2-1) \sum_{u=0}^{p} \{(-1)^{u+1} \sum_{m=k_u+1}^{k_{u+1}-1} \alpha_m t^{n-m-1}\}]$$
$$- [\beta_0 t^{n-1} + (-1)^{q+1}\beta_n t + (t^2+1) \sum_{s=1}^{q} (-1)^s \beta_{r_s} t^{n-r_s-1}$$
$$+ (t^2-1) \sum_{s=0}^{q} \{(-1)^{s+1} \sum_{v=r_s+1}^{r_{s+1}-1} \beta_v t^{n-v-1}\}] + |a_0| t^{n+1},$$

$$k_0 = r_0 = 0,$$
$$k_{p+1} = r_{q+1} = n.$$

Finally, in this paper, we prove two more results, (each, a generalization of Eneström-Kakeya theorem), with first one being somewhat similar to Theorem 2 and second one being somewhat similar to Theorem D.

THEOREM 3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n, with complex coefficients such that*

$$|arg\, a_j - \beta| \le \alpha \le \pi/2, \quad j = 0, 1, \ldots, n,$$

for ceratin real β *and for certain non-negative integers* $k_1, k_2, \ldots, k_p$ *and for certain* $t > 0$

$$|a_0| \le t|a_1| \le \ldots \le t^{k_1}|a_{k_1}| \ge t^{k_1+1}|a_{k_1+1}| \ge \ldots \ge t^{k_2}|a_{k_2}| \le t^{k_2+1}|a_{k_2+1}| \le \ldots \quad (1.8)$$

(with inequalities getting reversed at p indices $k_1, k_2, \ldots, k_p$ *in* (1.8) *and* $t^n|a_n|$, *being the last term in* (1.8)). *Then all zeros of* $p(z)$ *lie in*

$$R_3 \le |z| \le R_4,$$

where

$$R_3 = min\left(\frac{t^2|a_0|}{M_3}, t\right) = \frac{t^2|a_0|}{M_0} = \frac{t^2|a_0|}{M_3'}$$

$$R_4 = max\left(\frac{M_4}{|a_0|}, \frac{1}{t}\right),$$

$$M_3 = tM_3',$$

$$M_3' = -\{2\cos\alpha \sum_{m=1}^{p} (-1)^m |a_{k_m}| t^{k_m} + |a_0| + (-1)^{p+1}|a_n| t^n\}$$
$$+ 2\sin\alpha \sum_{j=0}^{n-1} |a_j| t^j + (-|a_0| + |a_n| t^n)\sin\alpha + |a_n| t^n,$$

$$M_2 = -[\cos\alpha\ \{(t^2-1) \sum_{m=0}^{p} ((-1)^{m+1} \sum_{s=k_m+1}^{k_{m+1}-1} |a_s| t^{n-s-1})\}$$
$$+ (t^2+1) \sum_{m=1}^{p} (-1)^m |a_{k_m}| t^{n-k_m-1} + |a_0| t^{n-1}(1+t^2)$$
$$+ (-1)^{p+1}|a_n| t] + \sin\alpha\ \{\sum_{j=1}^{n} (t|a_j| + |a_{j-1}|) t^{n-j}\} + |a_0| t^{n+1},$$

$$k_0 = 0, k_{p+1} = n.$$

THEOREM 4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$, *(Re* $a_j = \alpha_j$, *Im* $a_j = \beta_j$, *for* $j = 0, 1, 2, \ldots, n$), *be a polynomial of degree n such that, for certain real* β,

$$|arg\, a_j - \beta| \le \alpha \le \pi/2, \quad for\ j = 1, 2, \ldots, n,$$

and for certain $t\ (>0)$ *and* $k\ (0 < k \le n)$

$$|\alpha_0| \le t|\alpha_1| \le \ldots \le t^k|\alpha_k| \ge t^{k+1}|\alpha_{k+1}| \ge \ldots \ge t^n|\alpha_n|.$$

Then all zeros of $p(z)$ *lie in*

$$R_5 \le |z| \le R_6,$$

where

$$R_5 = \frac{t^2|a_0|}{M_5},$$

$$R_6 = max\left(\frac{M}{|a_n|}, \frac{1}{t}\right),$$

$$M_5 = t^{n+1}(|a_n| - |\alpha_n|) - t|\alpha_0| + \sum_{j=1}^{n} \big| t|\beta_j| - |\beta_{j-1}| \big| t^j$$

$$+ 2 \sin\alpha\, (t^{1/2} \sum_{j=1}^{n} |a_j a_{j-1}|^{1/2} t^j) + 2t^{k+1}|\alpha_k|,$$

$$M = \begin{cases} \left.\begin{array}{l} t^{n+1}|a_0| + t^{n-1}|\alpha_0| - t|\alpha_n| + (1-t^2) \sum_{j=1}^{n-1} |\alpha_j| t^{n-j-1} \\ + \sum_{j=1}^{n} \big| t|\beta_j| - |\beta_{j-1}| \big| t^{n-j} + 2 \sin\alpha\, (t^{1/2} \sum_{j=1}^{n} |a_j a_{j-1}|^{1/2} t^{n-j}), \end{array}\right\}, & k = 0, \\ \left.\begin{array}{l} t^{n+1}|a_0| - t^{n-1}|\alpha_0| - t|\alpha_n| + (t^2-1) \sum_{j=1}^{k-1} |\alpha_j| t^{n-j-1} \\ + (1+t^2) t^{n-k-1}|\alpha_k| + (1-t^2) \sum_{j=k+1}^{n-1} |\alpha_j| t^{n-j-1} \\ + \sum_{j=1}^{n} \big| t|\beta_j| - |\beta_{j-1}| \big| t^{n-j} + 2 \sin\alpha\, (t^{1/2} \sum_{j=1}^{n} |a_j a_{j-1}|^{1/2} t^{n-j}) \end{array}\right\}, & 1 \le k \le n-1, \\ \left.\begin{array}{l} t^{n+1}|a_0| - t^{n-1}|\alpha_0| + t|\alpha_n| + (t^2-1) \sum_{j=1}^{n-1} |\alpha_j| t^{n-j-1} \\ + \sum_{j=1}^{n} \big| t|\beta_j| - |\beta_{j-1}| \big| t^{n-j} + 2 \sin\alpha\, (t^{1/2} \sum_{j=1}^{n} |a_j a_{j-1}|^{1/2} t^{n-j}) \end{array}\right\}, & k = n. \end{cases}$$

2. LEMMAS

For the proofs of the theorems, we require the following lemmas.

LEMMA 1. *Let $f(z)$ be a polynomial of degree n, with*

$$M(r) = \max_{|z|=r} |f(z)|, \quad (r > 0).$$

Then

$$\frac{M(r_1)}{r_1^n} \ge \frac{M(r_2)}{r_2^n}, \quad 0 < r_1 < r_2.$$

Equality is attained only if the polynomial is of the form cz^n.

This lemma is due to Polya and Szegö [16, Part III: Problem no. 269].

LEMMA 2. *If a_j and a_{j-1} are two complex numbers with*

$$|\arg a_j - \beta| \le \alpha \le \pi/2,$$
$$|\arg a_{j-1} - \beta| \le \alpha \le \pi/2,$$

for certain real β, then

$$|a_j - a_{j-1}|^2 \le (|a_j| - |a_{j-1}|)^2 \cos^2\alpha + (|a_j| + |a_{j-1}|)^2 \sin^2\alpha.$$

This lemma is due to Govil and Rahman [9, Proof of Theorem 2].

LEMMA 3. *Under the same hypothesis, as in Lemma 2,*

$$|a_j - a_{j-1}| \le \big||a_j| - |a_{j-1}|\big| \cos\alpha + (|a_j| + |a_{j-1}|) \sin\alpha.$$

Proof of Lemma 3: It folows easily from Lemma 2.

LEMMA 4. *Under the same hypothesis, as is Lemma 2,*

$$|a_j - a_{j-1}|^2 \le (|a_j| - |a_{j-1}|)^2 + 4|a_j a_{j-1}| \sin^2\alpha.$$

This lemma is due to Jain [10, Proof of Theorem 2].

LEMMA 5. *If a_j ($Re\, a_j = \alpha_j$, $Im\, a_j = \beta_j$) and a_{j-1} ($Re\, a_{j-1} = \alpha_{j-1}$, $Im\, a_{j-1} = \beta_{j-1}$) are the two complex numbers with*

$$|\arg a_j - \beta| \le \alpha \le \pi/2,$$
$$|\arg a_{j-1} - \beta| \le \alpha \le \pi/2,$$

for certain real β, then

$$|a_j - a_{j-1}| \le \big||\alpha_j| - |\alpha_{j-1}|\big| + \big||\beta_j| - |\beta_{j-1}|\big| + 2\,|a_j \cdot a_{j-1}|^{1/2} \sin\alpha.$$

Proof of Lemma 5: We have

$$\begin{aligned}(|a_j| - |a_{j-1}|)^2 &= (|\alpha_j| - |\alpha_{j-1}|)^2 + (|\beta_j| - |\beta_{j-1}|)^2 \\ &\quad + 2(|\alpha_j \alpha_{j-1}| + |\beta_j \beta_{j-1}| - |a_j a_{j-1}|) \\ &\le (|\alpha_j| - |\alpha_{j-1}|)^2 + (|\beta_j| - |\beta_{j-1}|)^2,\end{aligned}$$

which, by Lemma 4, implies

$$|a_j - a_{j-1}|^2 \le (|\alpha_j| - |\alpha_{j-1}|)^2 + (|\beta_j| - |\beta_{j-1}|)^2 + 4|a_j a_{j-1}| \sin^2\alpha.$$

and Lemma 5 follows.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. We consider the polynomial

$$F(z) = (t_2 + z)(t_1 - z)\, p(z) \tag{3.1}$$

$$\begin{aligned} &= -a_n z^{z+2} + (a_n(t_1 - t_2) - a_{n-1}) z^{n-1} + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}) z^n + \dots \\ &\quad + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0) z^2 + (a_1 t_1 t_2 + a_0(t_1 - t_2)) z + a_0 t_1 t_2. \end{aligned} \tag{3.2}$$

Further, let

$$G(z) = z^{n+2} F(1/z) \tag{3.3}$$

$$\begin{aligned} &= -a_n + (a_n(t_1 - t_2) - a_{n-1}) z \\ &\quad + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}) z^2 + \dots + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0) z^n \\ &\quad + (a_1 t_1 t_2 + a_0(t_1 - t_2)) z^{n+1} + a_0 t_1 t_2 z^{n+2}, \end{aligned} \tag{3.4}$$

$$= \psi(z) + a_0 t_1 t_2 z^{n+2}, \text{ say.} \tag{3.5}$$

We now have on $|z| = 1/t_1$,

$$|\psi(z)| \le |a_n| + (a_n(t_1 - t_2) - a_{n-1})\frac{1}{t_1}$$

$$+ (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})\frac{1}{t_1^2} + \dots$$

$$+ (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0)\frac{1}{t_1^n} + (a_1 t_1 t_2 + a_0(t_1 - t_2))\frac{1}{t_1^{n+1}} \text{ (by (1.1) and (1.2))}$$

$$= |a_n| + a_n - a_0\frac{t_2}{t_1^{n+1}},$$

and therefore, by Lemma 1, we have, for $|z| \ge 1/t_1$

$$|\psi(z)| \le \left(|a_n| + a_n - a_0\frac{t_2}{t_1^{n+1}}\right)|z|^{n+1}t_1^{n+1},$$

which, by (3.5), helps us to write

$$|G(z)| \ge |a_0|t_1t_2|z|^{n+2} - \left(|a_n| + a_n - a_0\frac{t_2}{t_1^{n+1}}\right)|z|^{n+1}t_1^{n+1}, \quad |z| \ge 1/t_1$$

$$> 0,$$

if

$$|z| > \max\left\{\frac{|a_n|t_1^{n+1} + a_n t_1^{n+1} - a_0 t_2}{|a_0|t_1t_2}, \frac{1}{t_1}\right\}.$$

Hence $F(z)$ and therefore $p(z)$ has no zeros in

$$|z| < \min\left\{\frac{|a_0|t_1t_2}{|a_n|t_1^{n+1} + a_n t_1^{n+1} - a_0 t_2}, t_1\right\}. \tag{3.6}$$

Again, by (3.4), we have

$$G(z) = -a_n + \phi(z), \text{ say.} \tag{3.7}$$

We now have on $|z| = 1/t_1$

$$|\phi(z)| \le a_n - a_0\frac{t_2}{t_1^{n+1}} + |a_0|\frac{t_2}{t_1^{n+1}} \quad \text{(by (1.1) and (1.2))} \tag{3.8}$$

which, by Schwarz's lemma, helps us to write

$$|\phi(z)| \le \left(a_n - a_0\frac{t_2}{t_1^{n+1}} + |a_0|\frac{t_2}{t_1^{n+1}}\right)|z|t_1, \quad |z| \le 1/t_1,$$

and therefore, by (3.7), we have

$$|G(z)| \ge |a_n| - \left(a_n - a_0\frac{t_2}{t_1^{n+1}} + |a_0|\frac{t_2}{t_1^{n+1}}\right)|z|t_1, \quad |z| \le 1/t_1$$

$$> 0,$$

if

$$|z| < \min\left\{\frac{|a_n|}{a_n - a_0\frac{t_2}{t_1^{n+1}} + |a_0|\frac{t_2}{t_1^{n+1}}}, \frac{1}{t_1}\right\}:$$

Hence $F(z)$ and therefore $p(z)$ has no zeros in

$$|z| > \max\left\{\frac{a_n - a_0\frac{t_2}{t_1^{n+1}} + |a_0|\frac{t_2}{t_1^{n+1}}}{|a_n|}, t_1\right\}.$$

Theorem 1 now follows, by using (3.7).

Proof of Theorem 2: It is similar to the proof of the result [6, Theorem], with two changes:

(i) $\sum_{j=1}^{n} |t\alpha_j - \alpha_{j-1}|t^j$ being broken into $(p+1)$ sums (corresponding to p integers $k_1, k_2, \dots, k_p$), instead of two sums (corresponding to one integer k),

(ii) $\sum_{j=1}^{n} |t\beta_j - \beta_{j-1}|t^j$ being broken into $(q+1)$ sums (corresponding to q integers $r_1, r_2, \dots, r_q$), instead of two sums (corresponding to one integer r), and so we omit the details.

Proof of Theorem 3: It is also similar to the proof of the result [6, Theorem], with two changes:

(i) inequality (obtainable by Lemma 3)

$$|ta_j - a_{j-1}| \le (|t|a_j| - |a_{j-1}||)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha,$$

instead of the inequality

$$|ta_j - a_{j-1}| \le |t\alpha_j - \alpha_{j-1}| + |t\beta_j - \beta_{j-1}|,$$

(ii) $\sum_{j=1}^{n} |t|a_j| - |a_{j-1}||t^j$ being broken into $(p+1)$ sums (corresponding to p integers $k_1, k_2, \dots, k_p$), instead of two sums (corresponding to one integer k), and so we omit the details.

Proof of Theorem 4: It is also similar to the proof of result [6, Theorem], with two changes:

(i) inequality (obtainable by Lemma 5)

$$|ta_j - a_{j-1}| \le |t|\alpha_j| - |\alpha_{j-1}|| + |t|\beta_j| - |\beta_{j-1}|| + 2t^{1/2}|a_j a_{j-1}|^{1/2} \sin \alpha,$$

instead of the inequality

$$|ta_j - a_{j-1}| \le |t\alpha_j - \alpha_{j-1}| + |t\beta_j - \beta_{j-1}|,$$

(ii) no break up of $\sum_{j=1}^{n} |t|\beta_j| - |\beta_{j-1}||t^j$

and so we omit the details.

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