Data-flow Analysis / Abstract Interpretation

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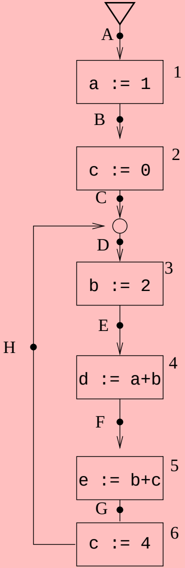
What is data-flow analysis

- "Computing 'safe' approximations to the set of values / behaviours arising dynamically at run time, statically or at compile time."
- Typically used by compiler writers to optimize running time of compiled code.
  - Constant propogation: Is the value of a variable constant at a particular program location.
  - Replace `x := y + z` by `x := 17` during compilation.
- More recent interest by verification community (starting with Cousot-Cousot 1977).
- Ideas used in SLAM tool to verify properties ("lock-unlock protocol is respected") of device driver code.

Constant Propogation Example

A variable x has constant value c at a program point N if along every execution the value of x at N is c.

Example: At program point G, constants are $R_G = \{(a,1),(b,2),(d,3)\}$.

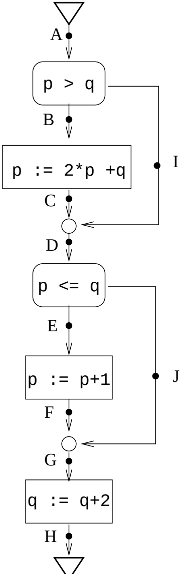


Overview of data-flow analysis

- Informal intro and motivation
- Lattices
- Data-flow analysis more formally
- Kildall's algo for computing over-approximation of JOP.
- Knaster-Tarski Fixpoint Theorem
- Correctness of Kildall's algo (computes the least solution to equations).

Data-flow analysis as approximation of collecting semantics

- Collecting semantics of a program: For each program point N, the set of states the program could be in at point N.



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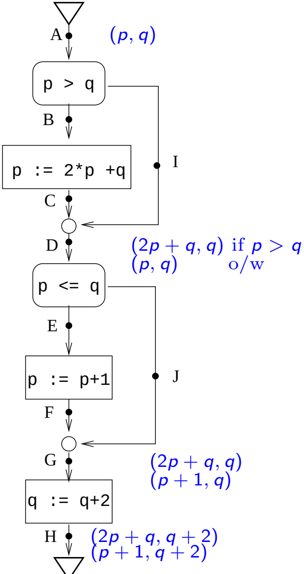

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- Example: Parity-based abstract interpretation.
- Abstract values: o, e, oe
- States represented:
  $o \mapsto \{1, 3, 5, \ldots\}$,
  $e \mapsto \{0, 2, 4, \ldots\}$,
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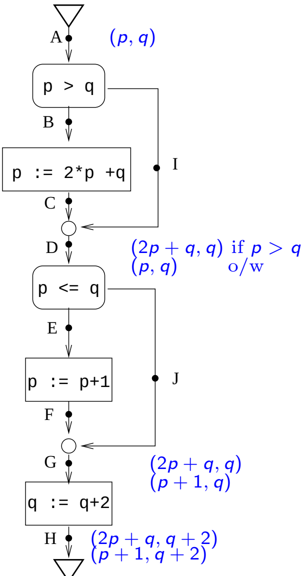
- So $(o, e) \mapsto \{1, 3, 5, \ldots\} \times \{0, 2, 4, \ldots\}$.



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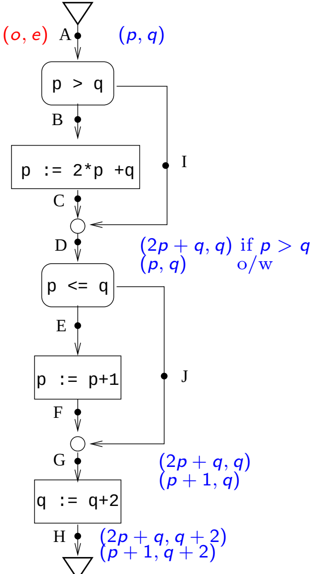

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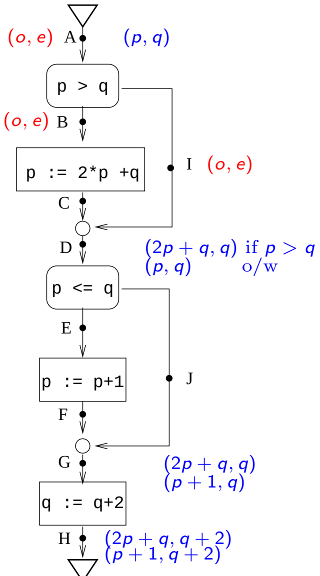

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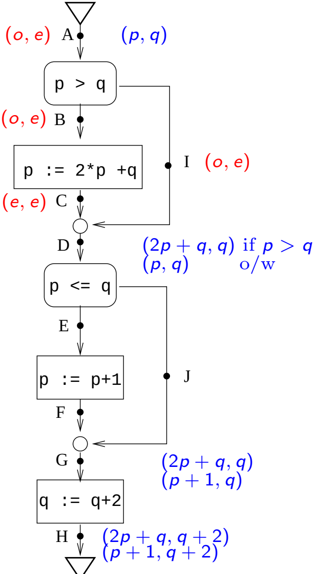

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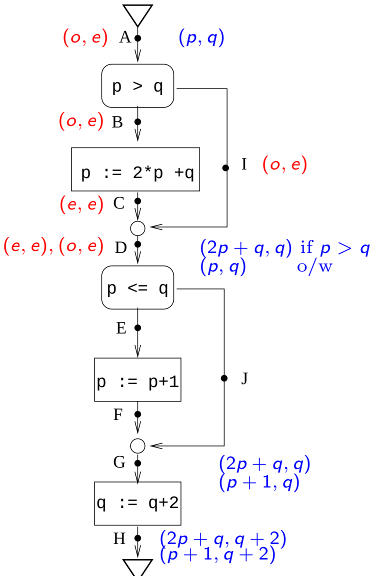

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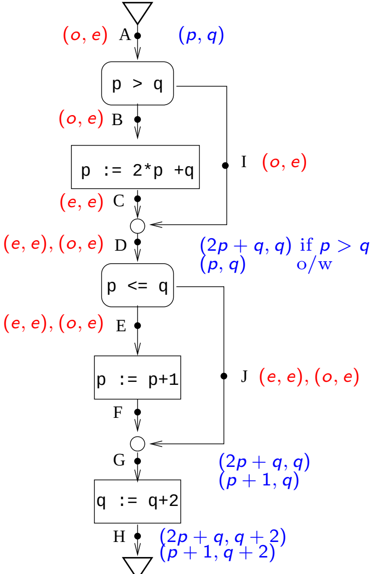

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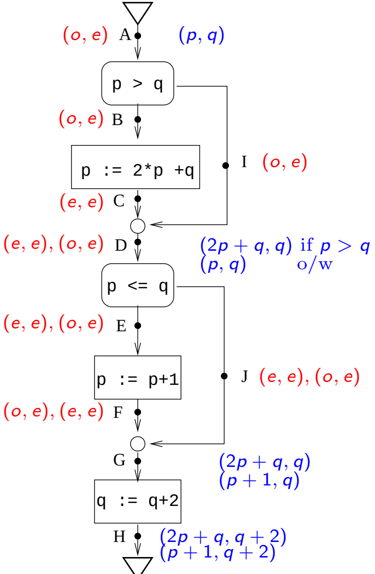

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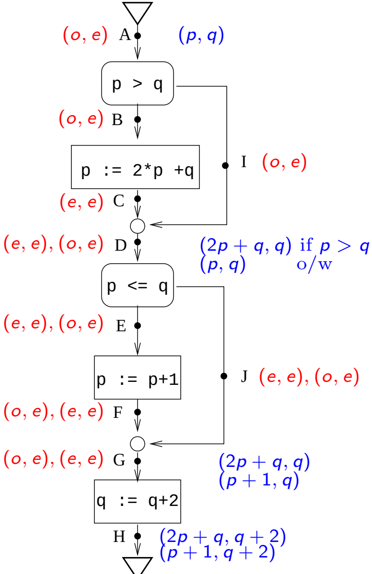

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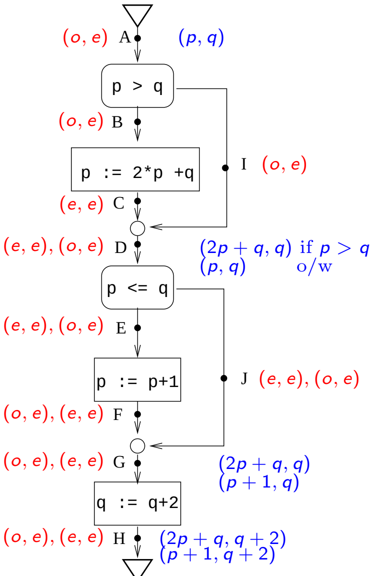
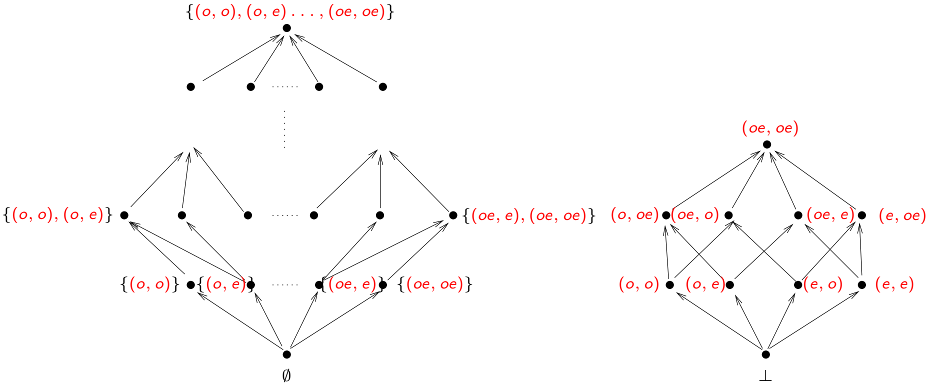
- So $(o, e) \mapsto \{1, 3, 5, \ldots\} \times \{0, 2, 4, \ldots\}$.
- Abstract states at H represents $\mathbb{N} \times 2\mathbb{N}$, which is a safe approx



Why abstract data should have a "lattice" structure

- A natural subset lattice structure:



- ... and a more "efficient" but less-precise lattice.
- Ordering is "is more precise than".
- Take "join" or "least upper bound" of abstract states at a point.

Why transfer functions should be "monotonic"

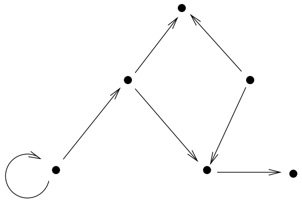
- More precise source state should lead to more precise target state.

Partial Orders

- A partially ordered set is a non-empty set D along with a partial order $\leq$.
  - $\leq$ is reflexive ($d \leq d$ for each $d \in D$)
  - $\leq$ is transitive ($d \leq d'$ and $d' \leq d''$ implies $d \leq d''$)
  - $\leq$ is anti-symmetric ($d \leq d'$ and $d' \leq d$ implies $d = d'$).

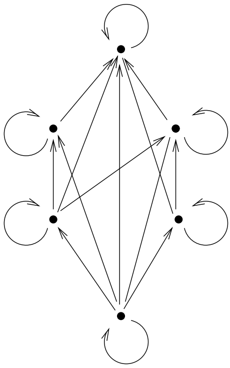
Binary relations as Graphs

We can view a binary relation on a set as a directed graph.

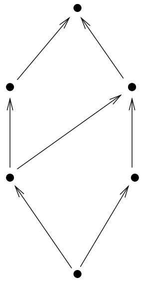


Partial Order as a graph

A partial order is then a special kind of directed graph:



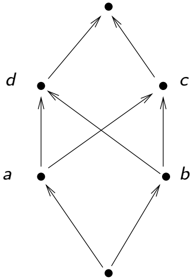
Graph representation



Hasse-diagram representation

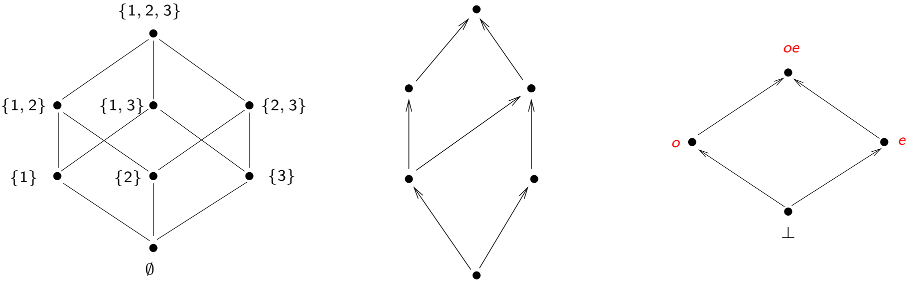
Upper bounds etc.

- An element $u \in D$ is an upper bound of a set of elements $X \subseteq D$, if $x \leq u$ for all $x \in X$.
- u is the least upper bound (or lub or join) of X if u is an upper bound for X, and for every upper bound y of X, we have $u \leq y$. We write $u = \bigsqcup X$.
- Similarly, $v = \bigsqcap X$ (v is the greatest lower bound or glb or meet of X).



Lattices

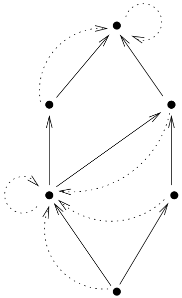
- A lattice is a partially order set in which every pair of elements has an lub and a glb.
- A complete lattice is a lattice in which every subset of elements has a lub and glb.



Question: Example of lattice which is not complete?

Monotonic functions

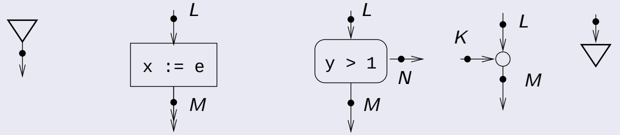
- A function $f : D \to D$ is monotonic or order-preserving if whenever $x \leq y$ we have $f(x) \leq f(y)$.



Data-flow / abstract-interpretation framework

Program are finite directed graphs with following nodes (statements):

Nodes or statements in a program



- Expressions:

$$e ::= c \mid x \mid e + e \mid e - e \mid e * e.$$

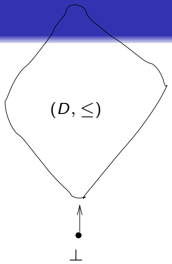
- Boolean expressions:

$$be ::= tt \mid ff \mid e \leq e \mid e = e \mid \neg be \mid be \vee be \mid be \wedge be.$$

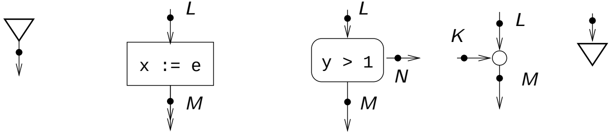
- Assume unique initial node I.

Data-flow framework contd.

- Complete lattice $L = (D, \leq)$.
- Add new bottom element to get $L_\perp = (D_\perp, \leq_\perp)$.



- Transfer function $f_{LM} : D_\perp \rightarrow D_\perp$ for each node and incoming edge L and outgoing edge M.



- We assume transfer functions are monotonic, and satisfy $f(\perp) = \perp$.
- Junction nodes have identity transfer function.

What we want to compute for a given program

- Path in a program: Sequence of connected edges or program points.
- Transfer functions extend to paths in program:

$$f_{ABCD} = f_{CD} \circ f_{BC} \circ f_{AB}.$$

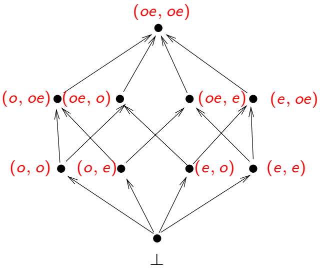
- For "infeasible" paths p, f_p will be $\lambda d.\bot$.
- Join over all paths (JOP) definition: For each program point N

$$d_N = \bigsqcup_{\text{paths } p \text{ from } I \text{ to } N} f_p(d_0).$$

where d_0 is a given initial value at entry node.

Example framework: parity interpretation

- Underlying lattice



- Transfer functions: for `x := e` node:

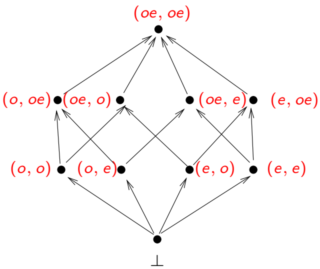
$$f_{MN}(s) = \begin{cases} s[x \mapsto o] & \text{if } [e]_s = o \\ s[x \mapsto e] & \text{if } [e]_s = e \\ s[x \mapsto oe] & \text{if } [e]_s = oe \end{cases}$$

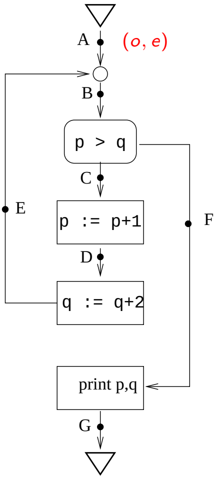
Kildall's algorithm to compute over-approximation of JOP

- Initialize data value at each program point to $\bot$, entry node to d_0.
- Mark data values at all nodes.
- Repeat while there is a marked value:
  - Choose a node M with marked value d_M, unmark it, and "propogate" it to successor nodes (i.e. for each successor node N, replace value at N by $f_{MN}(d_M) \sqcup d_N$).
  - Mark value at successor node if old value was marked, or new value larger than old value.
- Return data values at each point as over-approx of JOP.

Kildall's algo on parity interpretation example

Underlying lattice





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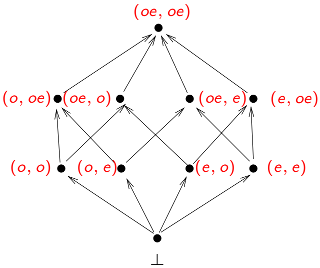

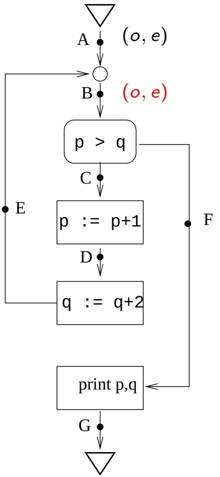

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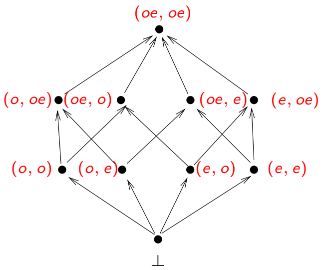

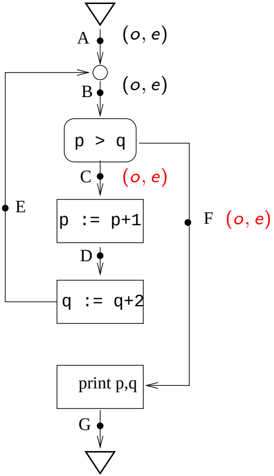

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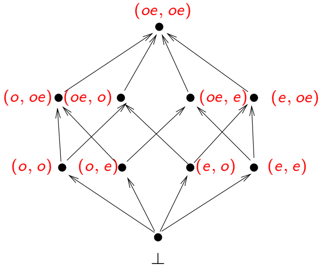

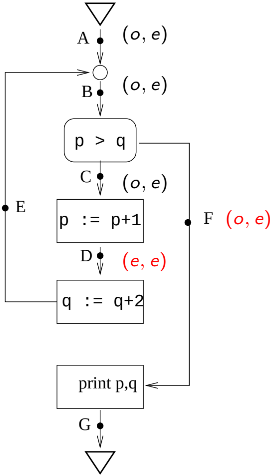

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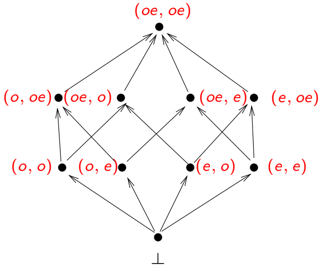

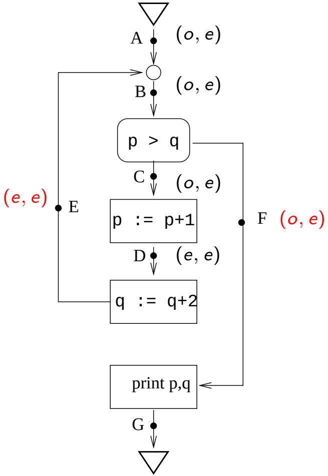

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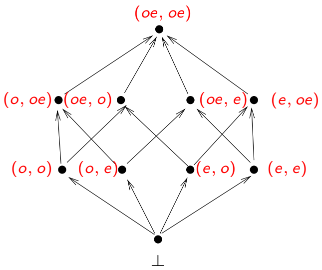

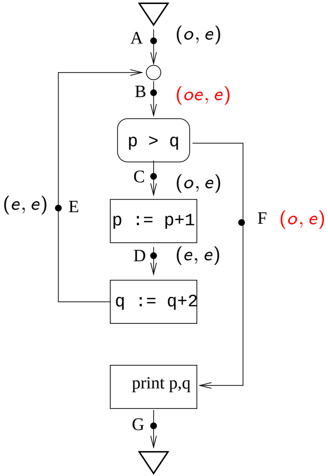

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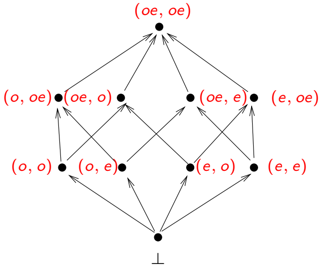

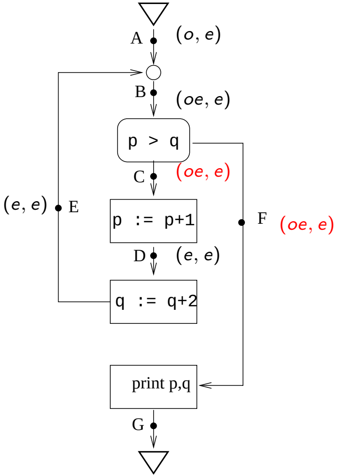

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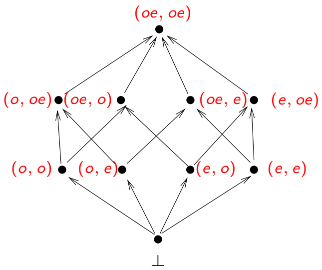

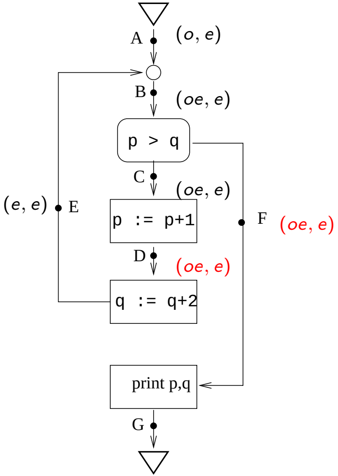

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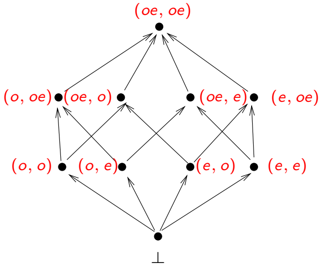

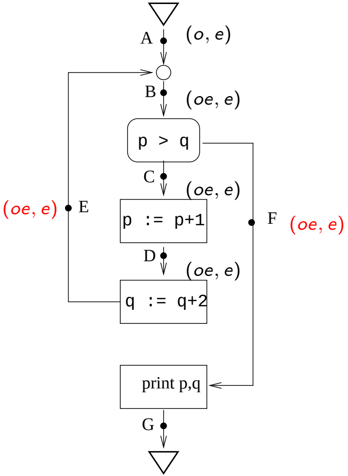

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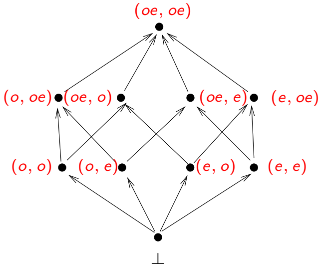

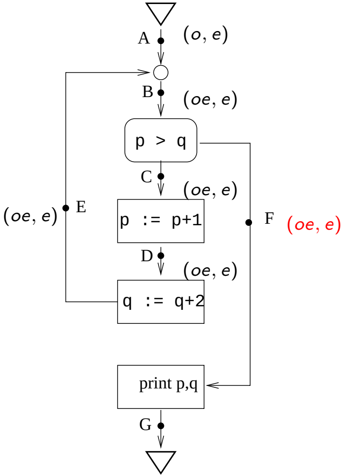

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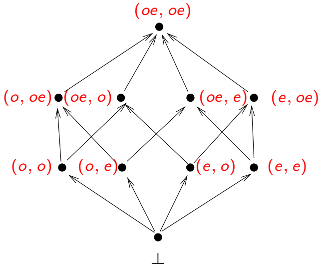

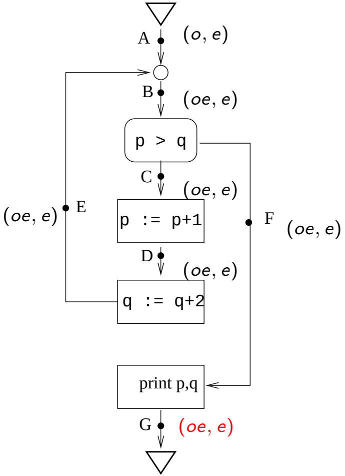

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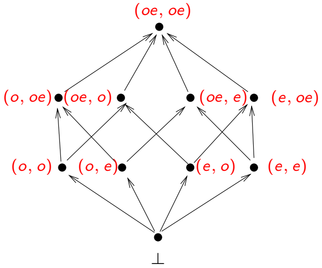
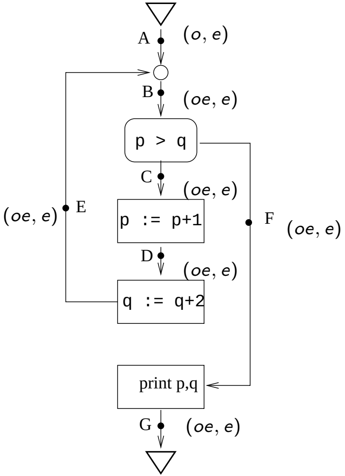
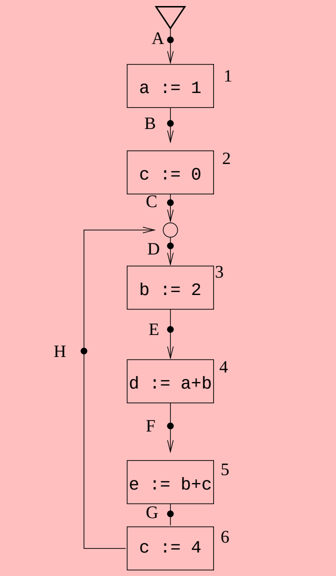




Another example analysis: Constant Propogation

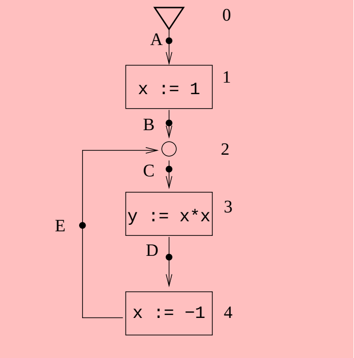
A variable x has constant value c at a program point N if along every execution the value of x at N is c.

Example: At program point G, constants are $R_G = \{(a,1),(b,2),(d,3)\}$.



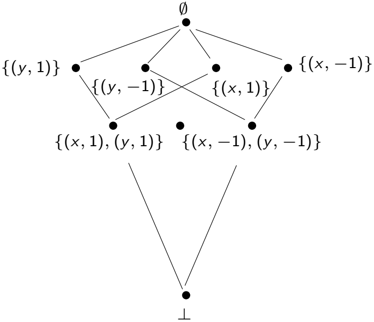
Another example program

ProgPt	Actual constant data
A	$\emptyset$
B	$(x, 1)$
C	$\emptyset$
D	$(y, 1)$
E	$(x, -1), (y, 1)$



Framework instance for CP

- Underlying lattice

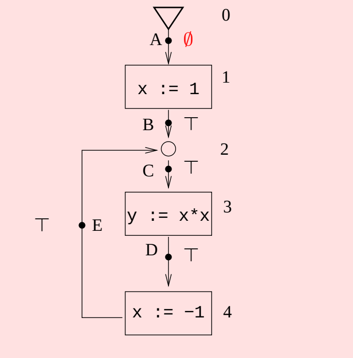


- Transfer function for assignment node n of the form $x := exp$.

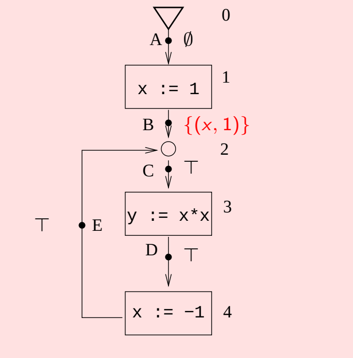
$$f_n(P) = \{(y, c) \mid y \neq x\} \cup \begin{cases} \{(x, d)\} & \text{if } [exp]_P = d \\ \emptyset & \text{otherwise.} \end{cases}$$

- Initial value at entry node: $\emptyset$.
- Transfer functions monotonic?

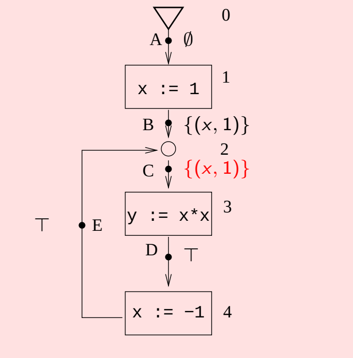
Kildall's algo on CP example: 1



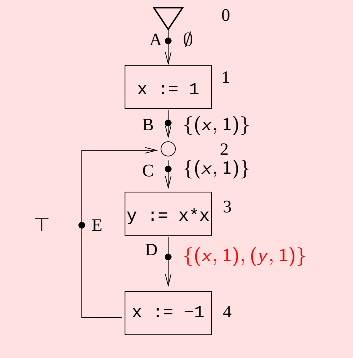
Kildall's algo on CP example: 2



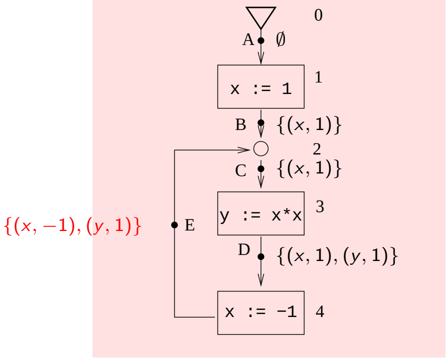
Kildall's algo on CP example: 3



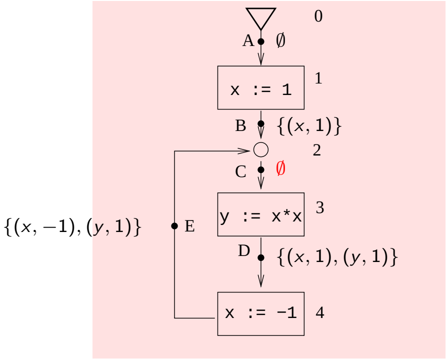
Kildall's algo on CP example: 4



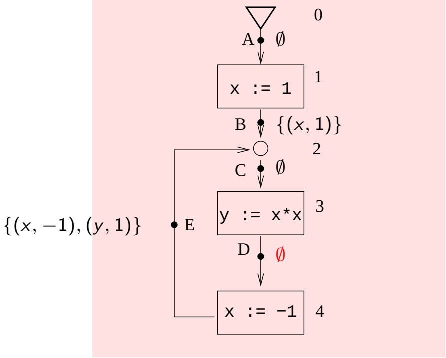
Kildall's algo on CP example: 5



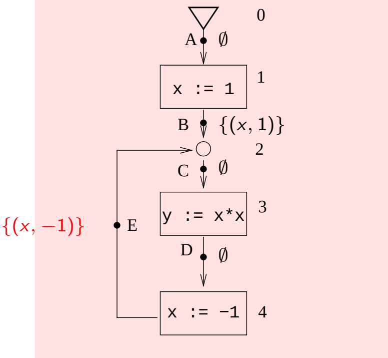
Kildall's algo on CP example: 6



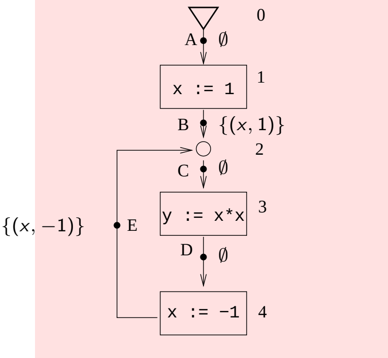
Kildall's algo on CP example: 7



Kildall's algo on CP example: 8

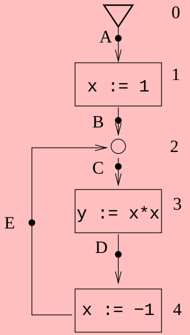


Kildall's algo on CP example: 9



Kildall's algo vs Actual Constant data

ProgPt	Actual data	Kildall's data
A	$\emptyset$	$\emptyset$
B	$(x, 1)$	$(x, 1)$
C	$\emptyset$	$\emptyset$
D	$(y, 1)$	$\emptyset$
E	$(x, -1), (y, 1)$	$(x, -1)$

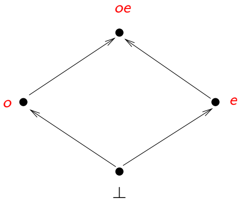


What Kildall's algo computes

- In general, computes an over-approximation of JOP.
- Always terminates if lattice has no infinite ascending chains.

More on lattices

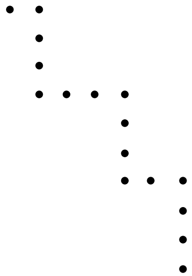
- A Chain in a partial order $(D, \leq)$ is a totally ordered subset of D.
- Ascending chain: $d_0 < d_1 < d_2 < \ldots$.
- Let $L = (D, \leq)$ be a complete lattice.
- The product lattice $\overline{L} = (D \times D, \leq')$ where $(d_1, d_2) \leq' (d_1', d_2')$ iff $d_1 \leq d_1'$ and $d_2 \leq d_2'$ is also a complete lattice.
- Exercise: compute product of parity lattice below with itself.



- Maximum ascending chain in $\overline{L} = L \times L$ is bounded by twice max ascending chain in L (*if* there is a max ascending chain in L).

Termination of Kildall's algo

- Let $\overline{d}_i$ be the vector of values after the i-th step of algo.
- Then after each step i, either number of marks decreases by 1 and $\overline{d}_{i+1} = \overline{d}_i$, or number of marks increase by 0 or 1 and $\overline{d}_{i+1} > \overline{d}_i$.
- Thus each $\overline{d}_i$ increases ($\geq$), and if it doesn't strictly increase we lose a mark.
- Thus maximum number of steps in algo is bounded by length of longest increasing chain in $\overline{L}$ * number of program points.



Viewing correctness

Extend f_n's to $\overline{f}$ over $\overline{D} = D \times \cdots \times D$ given by

$$\overline{f}(d_1, \ldots, d_k) = (\cdots, f_m(D_j), \cdots).$$

Then:

- $\overline{L} = (\overline{D}, \leq')$ is also a complete lattice.
- $\overline{f}$ is montonic on $\overline{L}$ if each f_n is.
- Set up equations Eq relating the data values at each program point.
- Least solution to Eq is same as LFP of functional $\overline{f}$ on lattice $\overline{L}$.
- If each f_n is distributive, then JOP $=$ LFP$(\overline{f})$.
- Otherwise, if f_n is only monotonic, JOP $\leq$ LFP$(\overline{f})$.
- Kildall's algo computes least solution to Eq, for monotone frameworks.
- Note this is a stronger claim than "Kildall's algo computes JOP for distributive frameworks".

Induced Equations

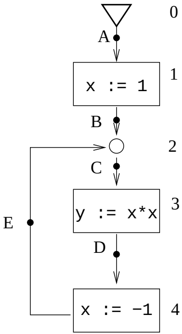
Framework induces natural data-flow equations:

x_E	$=$	e	for an entry node E
x_N	$=$	$f_n(x_M)$	for an assignment node n with incoming point M and outgoing point N
x_N	$=$	$X_L \sqcup X_M$	for a junction node with incoming points L,M and outgoing N.
	$\cdots$		etc.

Equations for CP example

Equations induced by CP analysis:

$$
\begin{aligned}
x_A &= \emptyset \\
x_B &= f_1(x_A) \\
x_C &= x_B \sqcup x_E \\
x_D &= f_3(x_C) \\
x_E &= f_4(x_D).
\end{aligned}
$$



Equations for CP example

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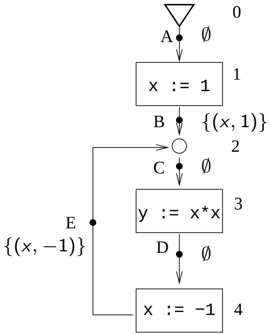
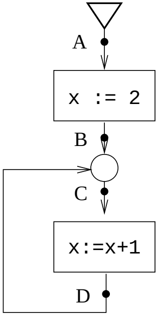
x_E &= f_4(x_D).
\end{aligned}
$$

Values computed by Kildall are a solution to these equations.



Exercise: Give 2 solutions to equations induced for this program

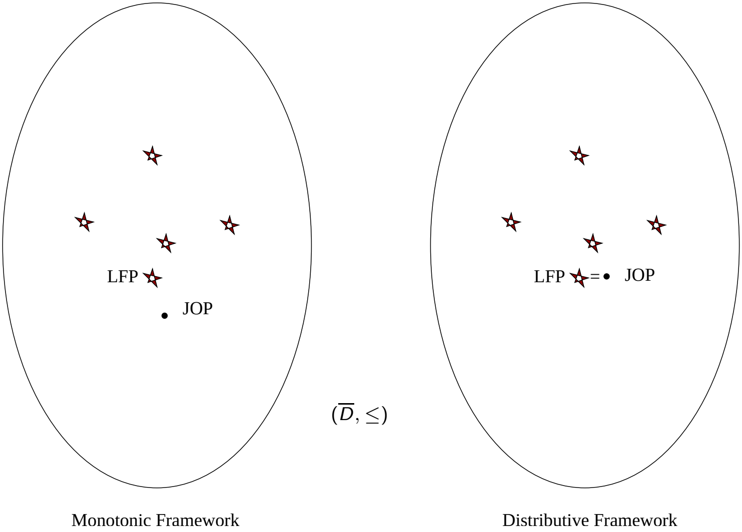
- Use collecting semantics with concrete stores in $\{x\} \to \mathbb{Z}$.
- Write down induced equations.
- Give two different solutions to the equations.



Natural ordering on solutions

- Consider "vectorised" lattice $\overline{D} = (D^k, \leq')$ (similar to product lattice $L \times L$).
- Each solution is a point in this vectorised lattice
- We will see that these solutions form a complete lattice, with least and greatest element.
- This is the least solution we mean.
- In fact a solution is a "fixpoint" of a natural function $\overline{f}$ induced by transfer functions for each node.

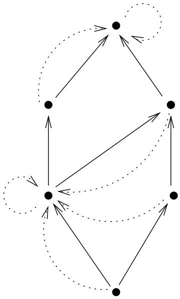
Correctness



Kildall's algo always computes LFP.

Knaster-Tarski fixpoint theorem for lattices

- A lattice is a partially order set in which every pair of elements has an lub and a glb.
- A complete lattice is a lattice in which every subset of elements has a lub and glb.
- A function $f : D \rightarrow D$ is monotonic or order-preserving if whenever $x \leq y$ we have $f(x) \leq f(y)$.
- A fixpoint of a function $f : D \rightarrow D$ is an element $x \in D$ such that $f(x) = x$.
- A pre-fixpoint of f is an element x such that $x \leq f(x)$.



Knaster-Tarski Fixpoint Theorem

Theorem (Knaster-Tarski)

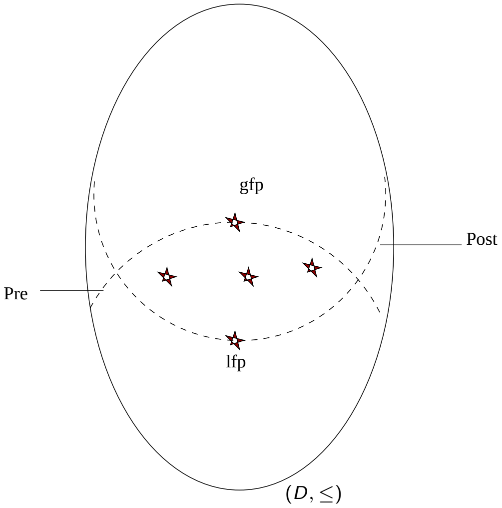
Let $(D, \leq)$ be a complete lattice, and $f : D \rightarrow D$ a monotonic function on $(D, \leq)$. Then:

(a) *f has at least one fixpoint.*

(b) *The set of fixpoints P of f itself forms a complete lattice under $\leq$.*

(c) *The least fixpoint of f coincides with the glb of the set of postfixpoints of f, and the greatest fixpoint of f coincides with the lub of the prefixpoints of f.*

Fixpoints of f



Proof of Knaster-Tarski theorem

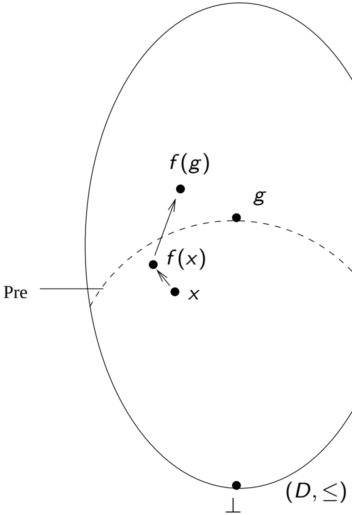
(a) $g = \bigsqcup \mathit{Pre}$ is a fixpoint of f.

(b) g is the greatest fixpoint of f.

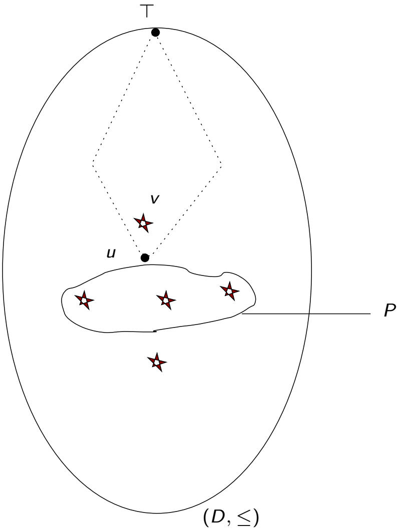
(c) Similarly $l = \bigsqcap \mathit{Post}$ is the least fixpoint of f.

(d) Let P be the set of fixpoints of f. Then $(P, \leq)$ is a *complete* lattice.

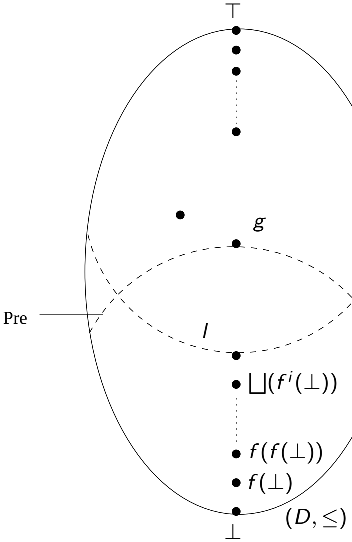
Proof of K-T theorem: (a)



Proof of K-T theorem: (d)



Computing lfp's and gfp's



Computing lfp's and gfp's

- "Ascending Chain Condition": No infinite ascending chains, or
- Continuity:
  - $X \subseteq D$ is *directed* if every finite subset of X has an upper bound in X.
  - f on $(D, \leq)$ is *continuous* if for every directed subset X of D we have $f(\bigsqcup X) = \bigsqcup(f(X))$.

Then

$$lfp(f) = \bigsqcup(f^n(\bot)).$$

A more general condition

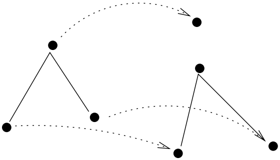
- A complete partial order (cpo) is a partial order in which every ascending chain has an lub.
- A pointed cpo is one which has a least element $\bot$.
- Let $(D, \leq)$ be a cpo. A function $f : D \to D$ is continuous if for any ascending chain X in D, $f(\bigsqcup X) = \bigsqcup(f(X))$.

Fact

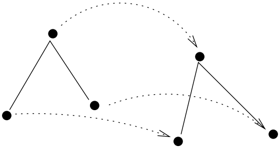
If f is a continuous function on a pointed cpo $(D, \leq)$ then f has a least fixpoint and

$$lfp(f) = \bigsqcup_{i \geq 0}(f^i(\bot)).$$

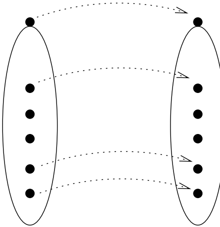
Monotonicity, distributivity, and continuity



Monotonic



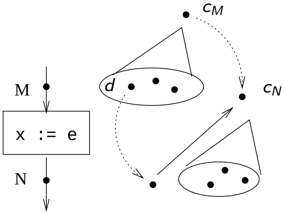
Distributive



Continuous

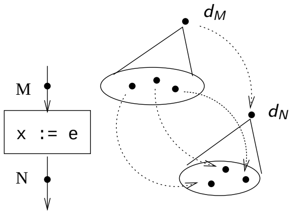
Back to Kildall: JOP $\leq$ LFP for monotone framework

- We show JOP $\leq \overline{c}$, for any FP $\overline{c}$.
- JOP $= \bigsqcup_{i \geq 0} \mathrm{JOP}_i$, where $\mathrm{JOP}_i = \bigsqcup_{\mathrm{paths}\ p, |p| \leq i} f_p(e)$.
- Claim: $\mathrm{JOP}_i \leq \overline{c}$ for any fixpoint $\overline{c}$.
    - By induction on i: Base case immediate.
    - Assume $\mathrm{JOP}_i \leq \overline{c}$, and consider JOP_{i+1}



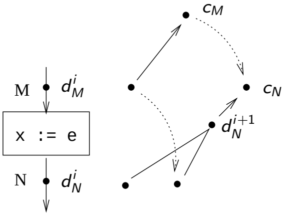
Correctness: JOP = LFP for finite distributive framework

- JOP = LFP for distributed framework, finite lattice.
- Enough to show that JOP is a fixpoint of $\overline{f}$.



What Kildall's algo computes (ctd)

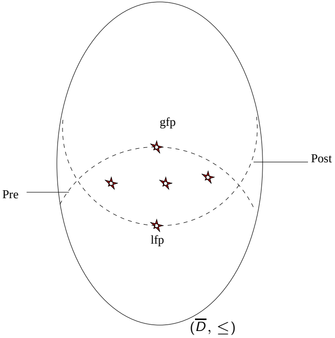
- Values at each step are bounded above by any fixed point $\overline{c}$.



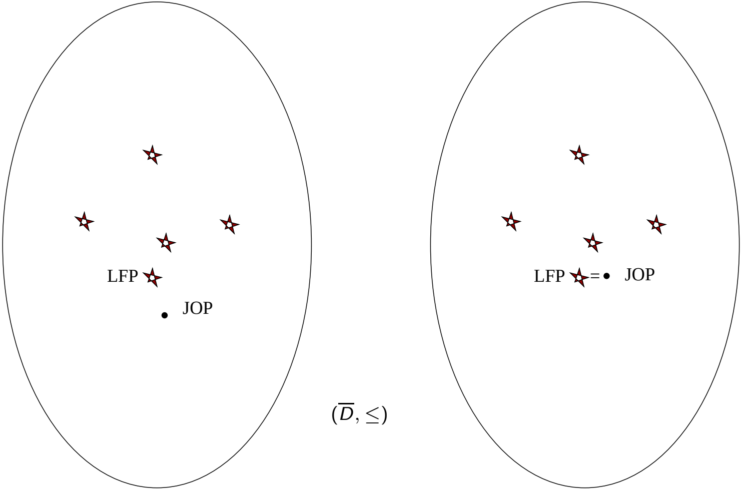
- Thus it follows that $\overline{d} \leq \overline{l}$ where $\overline{l}$ is LFP of $\overline{f}$.

What Kildall's algo computes (ctd)

- Sufficient now to show that $\overline{d} \geq \overline{l}$.
  - Suffices to show that $\overline{d}$ is such that $\overline{d} \geq \overline{f}(\overline{d})$ (i.e. $\overline{d}$ is a postfixpoint of $\overline{f}$)
    - We observe that if a value d_M^i was unmarked at some step in the algo, its value would have been propogated.
    - Thus, in particular, $d_N \geq f_{MN}(d_M)$, since d_M would have been propogated.
  - By Knaster-Tarski theorem, $\overline{l} = glb(Post)$, and hence $\overline{d} \geq \overline{l}$.



Correctness



Monotonic Framework

Distributive Framework

Kildall's algo always computes LFP.

Back to Constant Propogation

- f_n^{CP} is monotonic
- f_n^{CP} is *not* distributive.
  - Consider node n with statement $y := x * x$, and abstract values $d_1 = \{(x, 1)\}$ and $d_2 = \{(x, -1)\}$.
  - $f_n(d_1 \sqcup d_2) = \top$
  - $f_n(d_1) \sqcup f_n(d_2) = \{(y, 1)\}$.

Why computing JOP for CP is undecidable

- Post Correspondence Problem (PCP): Given strings $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, is there a string $w = u_{i_1} u_{i_2} \cdots u_{i_l}$ such that $w = v_{i_1} v_{i_2} \cdots v_{i_l}$, with $i_1 = 1$.
- Consider program for which computing JOP for Constant Propogation implies solution to PCP.

```
while (*) {
  if(*) {
    x := x * u_1;
    y := y * v_1;
  }
  ......
  if(*) {
    x := x * u_n;
    y := y * v_n;
  }
}
if (x == y) z := 1 else z := -1;
```